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REAL INTERPOLATION WITH VARIABLE EXPONENT

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We present the real interpolation with variable exponent and we prove the basic properties in analogy to the classical real interpolation. More precisely, we prove that under some additional conditions, this method can be reduced to the case of fixed exponent. An application, we give the real interpolation of variable Besov and Lorentz spaces as introduced recently in Almeida and Hästö (J. Funct. Anal. 258 (5) 1628–2655, 2010) and L. Ephremidze et al. (Fract. Calc. Appl. Anal. 11 (4) (2008), 407–420).

1. Introduction

It is well known that real interpolation play an important role in several different areas, especially for modern analysis and its theory started early in 1960's by J–L. Lions and J. Peetre. There are two ways for introducing the real interpolation method. The first is the K–method and the second is the J–method. But the spaces generated by the K– and J–methods are the same. For general literature on real interpolation we refer to [4], [5], [18] and references therein.

In recent years, there has been growing interest in generalizing classical spaces such as Lebesgue spaces, Sobolev spaces, Besov spaces and Triebel–Lizorkin spaces to the case with either variable integrability or variable smoothness. For instance, they appear in the study of variable exponent Riesz and

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Wolff potentials, see [3] where the authors use the real interpolation between the spaces $L^{p(\cdot)}$ and L^1. The motivation for the increasing interest in such spaces comes not only from theoretical purposes, but also from applications to fluid dynamics, image restoration and PDE with non-standard growth conditions.

From these in this paper we present a variable version of real interpolation. First we study the variable version of K–method, where we present some equivalent norms for the space generated by this method and we prove their basic properties in analogy to the fixed exponent. Secondly, we present the same analysis for the variable version of J–method and we prove the first main statement of this paper. That is, under some additional conditions the spaces generated by the K– and J–methods are the same. Since the reiteration theorem is one of the most important general results in interpolation theory, we will give its proof. Finally, we study the real interpolation of variable exponent Besov and Lorentz spaces. Almost all of the material we present is due to [4] and [5]. Allowing the exponent is vary from point to point will raise extra difficulties which, in general, are overcome by imposing regularity assumptions on this exponent.

2. Preliminaries

As usual, we denote by $\mathbb{R}^n$ the n-dimensional real Euclidean space, $\mathbb{N}$ the collection of all natural numbers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. The letter $\mathbb{Z}$ stands for the set of all integer numbers. The expression $f \lesssim g$ means that $f \leq c\, g$ for some independent constant c (and non-negative functions f and g), and $f \approx g$ means $f \lesssim g \lesssim f$.

By c we denote generic positive constants, which may have different values at different occurrences. Although the exact values of the constants are usually irrelevant for our purposes, sometimes we emphasize their dependence on certain parameters (e.g. c_p, or $c(p)$, means that c depends on p, etc.). Further notation will be properly introduced whenever needed.

We denote by $\mathcal{P}(\mathbb{R}^n)$ the set of all measurable functions p on $\mathbb{R}^n$ with range in $[1, \infty[$. We use the standard notation $p^- = \underset{x \in \mathbb{R}^n}{\text{ess–inf}}\, p(x)$ and $p^+ = \underset{x \in \mathbb{R}^n}{\text{ess–sup}}\, p(x)$.

The variable exponent modular is defined by $\rho_{p(\cdot)}(f) = \int_{\mathbb{R}^n} \rho_{p(x)}(|f(x)|)dx$, where $\rho_p(t) = t^p$. The variable exponent Lebesgue space $L^{p(\cdot)}$ consists of measurable functions f on $\mathbb{R}^n$ such that $\rho_{p(\cdot)}(\lambda f) < \infty$ for some $\lambda > 0$. We define the Luxemburg norm on this space by the formula $\|f\|_{p(\cdot)} = \inf\Big\{\lambda > 0 : \rho_{p(\cdot)}\Big(\frac{f}{\lambda}\Big) \leq 1\Big\}$. A useful property is that $\|f\|_{p(\cdot)} \leq 1$ if and only if $\rho_{p(\cdot)}(f) \leq 1$, see [8], Lemma 3.2.4.

We say that $g : \mathbb{R}^n \to \mathbb{R}$ is *locally* log-*Hölder continuous*, abbreviated $g \in C^{\log}_{\rm loc}(\mathbb{R}^n)$, if there exists $c_{\log}(g) > 0$ such that

$$|g(x) - g(y)| \le \frac{c_{\log}(g)}{\log(e + 1/|x-y|)}$$

for all $x, y \in \mathbb{R}^n$. If

$$|g(x) - g(0)| \le \frac{c_{\log}(g)}{\ln(e + 1/|x|)}$$

for all $x \in \mathbb{R}^n$, then we say that g is log-*Hölder continuous at the origin* (or has a log *decay at the origin*). We say that g satisfies the log-*Hölder decay condition*, if there exists $g_\infty \in \mathbb{R}$ and a constant $c_{\log} > 0$ such that

$$|g(x) - g_\infty| \le \frac{c_{\log}}{\log(e + |x|)}$$

for all $x \in \mathbb{R}^n$. We say that g is *globally*-log-*Hölder continuous*, abbreviated $g \in C^{\log}(\mathbb{R}^n)$, if it is locally log-Hölder continuous and satisfies the log-Hölder decay condition. The constants $c_{\log}(g)$ and $c_{\log}$ are called the *locally* log-*Hölder constant* and the log-*Hölder decay constant*, respectively. We note that all functions $g \in C^{\log}_{\rm loc}(\mathbb{R}^n)$ always belong to L^∞.

We refer to the recent monograph [6] for further properties, historical remarks and references on variable exponent spaces.

2.1. Technical lemmas

In this subsection we present some results which are useful for us. The following lemma is from [8].

Lemma 2.1. *Let $A \subset \mathbb{R}^n$ and $p \in \mathcal{P}(\mathbb{R}^n)$ with $p^- < \infty$. If $\eta = \rho_{p(\cdot)}(f\chi_A) > 0$ or $p^+ < \infty$, then*

$$\min\left\{\eta^{\frac{1}{p^-}}, \eta^{\frac{1}{p^+}}\right\} \le \|f\chi_A\|_{p(\cdot)} \le \max\left\{\eta^{\frac{1}{p^-}}, \eta^{\frac{1}{p^+}}\right\}.$$

The next lemma is a Hardy-type inequality which is easy to prove.

Lemma 2.2. *Let $0 < a < 1$ and $0 < q \le \infty$. Let $\{\varepsilon_k\}_k$ be a sequences of positive real numbers and denote $\delta_k = \sum_{j=-\infty}^{\infty} a^{|k-j|}\varepsilon_j$. Then there exists constant $c > 0$ depending only on a and q such that*

$$\Big(\sum_{k=-\infty}^{\infty} \delta_k^q\Big)^{1/q} \le c \,\Big(\sum_{k=-\infty}^{\infty} \varepsilon_k^q\Big)^{1/q}.$$

Putting

$$w(Q) = \int_Q w(x)\,dx,$$

we will make use of the following statement, see [9], Lemma 3.3 for $w := 1$.

Lemma 2.3. *Let $p \in \mathcal{P}(\mathbb{R})$ and w be a weight function on $\mathbb{R}$. Then, putting*

$$p_Q^- = \underset{z\in Q}{\text{ess-inf}}\, p(z)$$

for a cube $Q = (a,b) \subset \mathbb{R}$ with $0 < a < b < \infty$, we have the following inequality:

$$\begin{aligned}
&\left(\frac{\gamma_m}{w(Q)} \int_Q |f(y)|\, w(y)\,dy\right)^{p(x)} \\
\leq & c\ \max\left(1, (w(Q))^{1-\frac{p(x)}{p^-}}\right) \frac{1}{w(Q)} \int_Q \phi(y) w(y)\,dy \\
&+ \frac{c\ \omega(m,b)}{w(Q)} \int_Q g(x,y) w(y)\,dy
\end{aligned}$$

for some positive constant $c > 0$, all $x \in Q$ and all $f \in L^{p(\cdot)}(w)$ with

$$\|f\|_{L^{p(\cdot)}(w)} \leq 1,$$

where we put

$$\omega(m,b) = \min(b^m, 1), \quad \phi(y) = |f(y)|^{p(y)}$$

and

$$g(x,y) = \left(e + \frac{1}{x}\right)^{-m} + \left(e + \frac{1}{y}\right)^{-m},$$

or

$$\omega(m,b) = \min(b^m, 1),\ \phi(y) = |f(y)|^{p(0)}$$

and

$$g(x,y) = \left(e + \frac{1}{x}\right)^{-m} \chi_{\{z\in Q: p(z)<p(0)\}}(x)$$

with $p \in \mathcal{P}(\mathbb{R})$ being log-*Hölder continuous at the origin where*

$$\gamma_m = e^{-4mc_{\log}(1/p)} \in (0,1)$$

for every $m > 0$. In addition, we have the same estimate, when

$$\omega(m,b) = 1, \quad \gamma_m = e^{-mc_{\log}}, \quad \phi(y) = |f(y)|^{p_\infty}$$

and

$$g(x,y) = (e+x)^{-m} \chi_{\{z\in Q: p(z)<p_\infty\}}(x)$$

with $p \in \mathcal{P}(\mathbb{R})$ satisfying the log-*Hölder decay condition.*

The proof of this lemma is given in [11]. The next lemma is the continuous version of Hardy-type inequality, see [7].

Lemma 2.4. *Let $s > 0$. Let $q \in \mathcal{P}(\mathbb{R})$ be* log-*Hölder continuous both at the origin and at infinity with $1 \le q^- \le q^+ < \infty$. Let $\{\varepsilon_t\}_t$ be a sequence of positive measurable functions. Let*

$$\eta_t = t^s \int_t^\infty \tau^{-s} \varepsilon_\tau \frac{d\tau}{\tau} \quad \text{and} \quad \delta_t = t^{-s} \int_0^t \tau^s \varepsilon_\tau \frac{d\tau}{\tau}.$$

Then there exists constant $c > 0$ depending only on s, q^-, $c_{\log}(q)$ and q^+ such that

$$\|\eta_t\|_{L^{q(\cdot)}((0,\infty),\frac{dt}{t})} + \|\delta_t\|_{L^{q(\cdot)}((0,\infty),\frac{dt}{t})} \lesssim \|\varepsilon_t\|_{L^{q(\cdot)}((0,\infty),\frac{dt}{t})}.$$

3. The K–Method

The fundamental notion of real interpolation is the K–functional, where it is due to J. Peetre.

Definition 3.1. Let A_0 and A_1 be Banach spaces over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. We shall say that A_0 and A_1 are compatible if there is a Hausdorff topological vector space Z such that

$$A_0, A_1 \hookrightarrow Z,$$

with continuous embeddings.

Let A_0 and A_1 be compatible. We will say that (A_0, A_1) is a compatible couple. Then we can form their sum $A_0 + A_1$ and their intersection $A_0 \cap A_1$. The sum consists of all $f \in Z$ such that we can write

$$f = f_0 + f_1$$

for some $f_0 \in A_0$ and $f_1 \in A_1$. Then $A_0 + A_1$ is a Banach space with norm defined by

$$\|f\|_{A_0+A_1} = \inf_{f=f_0+f_1} \left(\|f_0\|_{A_0} + \|f_1\|_{A_1}\right).$$

$A_0 \cap A_1$ is a Banach space with norm defined by

$$\|f\|_{A_0\cap A_1} = \max_{f=f_0+f_1} \left(\|f_0\|_{A_0}, \|f_1\|_{A_1}\right).$$

Let (A_0, A_1) be a compatible couple. With $t > 0$ fixed, put

$$K(t, f; A_0, A_1) = \inf_{f=f_0+f_1} \left(\|f_0\|_{A_0} + t\,\|f_1\|_{A_1}\right), \quad f \in A_0 + A_1,$$

is the K–functional. For any $f \in A_0 + A_1$, $K(t, f; A_0, A_1)$ is a positive, increasing and concave function of t. In particular

$$K(t, f; A_0, A_1) \leq \max(1, \frac{t}{s}) K(s, f; A_0, A_1), \quad s, t > 0. \tag{1}$$

If there is no danger of confusion, we shall write $K(t, f) = K(t, f; A_0, A_1)$.

Definition 3.2. Let $\theta \in (0,1)$ and $q \in \mathcal{P}(\mathbb{R})$. Let (A_0, A_1) be a compatible couple. The space $(A_0, A_1)_{\theta, q(\cdot)}$ consists of all f in $A_0 + A_1$ for which the functional

$$\|f\|_{(A_0,A_1)_{\theta,q(\cdot)}} = \left\| t^{-\theta} K(t,f) \right\|_{L^{q(\cdot)}((0,\infty), \frac{dt}{t})}$$

is finite.

Definition 3.3. Let $\theta \in [0,1]$. Let (A_0, A_1) be a compatible couple. The space $(A_0, A_1)_{\theta,\infty}$ consists of all f in $A_0 + A_1$ for which the functional

$$\|f\|_{(A_0,A_1)_{\theta,\infty}} = \sup_{t>0} t^{-\theta} K(t,f)$$

is finite.

In the next lemma we prove that the first definition can be given in discrete version, where we need additional assumptions on q.

Lemma 3.4. *Let (A_0, A_1) be a compatible couple and $f \in A_0 + A_1$. Let $\theta \in (0,1)$, $f \in (A_0, A_1)_{\theta,q(\cdot)}$ and we put $\alpha_v = K(2^v, f)$, $v \in \mathbb{Z}$. Let $q \in \mathcal{P}(\mathbb{R})$ be log-Hölder continuous both at the origin and at infinity. Then*

$$\|f\|_{(A_0,A_1)_{\theta,q(\cdot)}} \approx \Big(\sum_{v=-\infty}^{0} 2^{-v\theta q(0)} \alpha_v^{q(0)} \Big)^{\frac{1}{q(0)}} + \Big(\sum_{v=1}^{\infty} 2^{-v\theta q_\infty} \alpha_v^{q_\infty} \Big)^{\frac{1}{q_\infty}}.$$

Moreover,

$$\|f\|_{(A_0,A_1)_{\theta,q(\cdot)}} \approx \Big(\int_0^1 t^{-\theta q(0)} K(t,f)^{q(0)} \frac{dt}{t} \Big)^{\frac{1}{q(0)}} + \Big(\int_1^\infty t^{-\theta q_\infty} K(t,f)^{q_\infty} \frac{dt}{t} \Big)^{\frac{1}{q_\infty}}.$$

Proof. We will do the proof in two steps and we need only to prove the first statement.

Step 1. Let us prove that

$$S = \Big(\sum_{v=-\infty}^{0} 2^{-v\theta q(0)} \alpha_v^{q(0)} \Big)^{\frac{1}{q(0)}} + \Big(\sum_{v=1}^{\infty} 2^{-v\theta q_\infty} \alpha_v^{q_\infty} \Big)^{\frac{1}{q_\infty}} \lesssim \|f\|_{(A_0,A_1)_{\theta,q(\cdot)}}. \tag{2}$$

By scaling argument, we need only to prove that

$$\sum_{v=1}^{\infty} 2^{-v\theta q_\infty}\big(\alpha_v\big)^{q_\infty} \lesssim 1 \quad \text{and} \quad \sum_{v=-\infty}^{0} 2^{-v\theta q(0)}\big(\alpha_v\big)^{q(0)} \lesssim 1$$

for any $f \in (A_0,A_1)_{\theta,q(\cdot)}$ with $\|f\|_{(A_0,A_1)_{\theta,q(\cdot)}} \leq 1$. To prove the first estimate we need to prove that

$$2^{-v\theta q_\infty}\big(\alpha_v\big)^{q_\infty} \leq \int_{2^{v-1}}^{2^v} \big(t^{-\theta}K(t,f)\big)^{q(t)}\frac{dt}{t} + 2^{-v} = \delta$$

for any $v \in \mathbb{N}$. This claim can be reformulated as showing that

$$\big(\delta^{-\frac{1}{q_\infty}} 2^{-v\theta}\alpha_v\big)^{q_\infty} = \Big(\frac{1}{\log 2}\int_{2^{v-1}}^{2^v} \delta^{-\frac{1}{q_\infty}} 2^{-v\theta}\alpha_v \frac{d\tau}{\tau}\Big)^{q_\infty} \lesssim 1.$$

Using the property (1), we find that

$$\int_{2^{v-1}}^{2^v} \delta^{-\frac{1}{q_\infty}} 2^{-v\theta}\alpha_v \frac{d\tau}{\tau} \lesssim \int_{2^{v-1}}^{2^v} \delta^{-\frac{1}{q_\infty}} \tau^{-\theta}K(\tau,f)\frac{d\tau}{\tau}.$$

By Lemma 2.3 the last expression with power $q(t)$ is bounded by

$$c\int_{2^{v-1}}^{2^v} \delta^{-\frac{q(\tau)}{q_\infty}} \big(\tau^{-\theta}K(\tau,f)\big)^{q(\tau)}\frac{d\tau}{\tau} + c$$

for any $t \in [2^{v-1},2^v]$. Since q is log-Hölder continuous at infinity, we find that

$$\delta^{-\frac{q(\tau)}{q_\infty}} \approx \delta^{-1}, \quad \tau \in [2^{v-1},2^v], \quad v \in \mathbb{N}. \tag{3}$$

Therefore, from the definition of δ, we find that

$$\int_{2^{v-1}}^{2^v} \delta^{-\frac{q(\tau)}{q_\infty}} \big(\tau^{-\theta}K(\tau,f)\big)^{q(\tau)}\frac{d\tau}{\tau} \lesssim 1.$$

Now, let us prove the second estimate. We need to show that

$$2^{-v\theta q(0)}\big(\alpha_v\big)^{q(0)} \lesssim \int_{2^{v-1}}^{2^v} \big(t^{-\theta}K(t,f)\big)^{q(t)}\frac{dt}{t} + 2^{v} = \delta$$

for any $v \leq 0$. This claim can be reformulated as showing that

$$\big(\delta^{-\frac{1}{q(0)}} 2^{-v\theta}\alpha_v\big)^{q(0)} = \Big(\frac{1}{\log 2}\int_{2^{v-1}}^{2^v} \delta^{-\frac{1}{q(0)}} 2^{-v\theta}\alpha_v \frac{d\tau}{\tau}\Big)^{q(0)} \lesssim 1.$$

The property (1), gives that

$$\int_{2^{v-1}}^{2^v} \delta^{-\frac{1}{q(0)}} 2^{-v\theta}\alpha_v \frac{d\tau}{\tau} \lesssim \int_{2^{v-1}}^{2^v} \delta^{-\frac{1}{q(0)}} \tau^{-\theta}K(\tau,f)\frac{d\tau}{\tau}.$$

Again by Lemma 2.3,

$$\Big(\int_{2^{v-1}}^{2^v}\delta^{-\frac{1}{q(0)}}\tau^{-\theta}K(\tau,f)\frac{d\tau}{\tau}\Big)^{q(t)}\lesssim\int_{2^{v-1}}^{2^v}\delta^{-\frac{q(\tau)}{q(0)}}\big(\tau^{-\theta}K(\tau,f)\big)^{q(\tau)}\frac{d\tau}{\tau}+1$$

for any $t\in[2^{v-1},2^v]$ and any $v\leq 0$. We use the logarithmic decay condition at origin of q to show that

$$\delta^{-\frac{q(\tau)}{q(0)}}\approx\delta^{-1},\quad \tau\in[2^{v-1},2^v],\quad v\leq 0.$$

Therefore, from the definition of δ, we find that

$$\int_{2^{v-1}}^{2^v}\delta^{-\frac{q(\tau)}{q(0)}}\big(\tau^{-\theta}K(\tau,f)\big)^{q(\tau)}\frac{d\tau}{\tau}\lesssim 1$$

for any $v\leq 0$. Hence, we proved (2).

Step 2. Let us prove that

$$\|f\|_{(A_0,A_1)_{\theta,q(\cdot)}}\lesssim S.$$

This claim can be reformulated as showing that

$$\int_0^\infty\big(t^{-\theta}K(t,\frac{f}{S})\big)^{q(t)}\frac{dt}{t}\lesssim 1.$$

Now our estimate clearly follows from the inequalities

$$\int_{2^{v-1}}^{2^v}\big(t^{-\theta}K(t,\frac{f}{S})\big)^{q(t)}\frac{dt}{t}\lesssim 2^{-v\theta q_\infty}\big(\frac{\alpha_v}{S}\big)^{q_\infty}+2^{-v}=\delta$$

for any $v\in\mathbb{N}$ and

$$\int_{2^{v-1}}^{2^v}\big(t^{-\theta}K(t,\frac{f}{S})\big)^{q(t)}\frac{dt}{t}\lesssim 2^{-v\theta q(0)}\big(\frac{\alpha_v}{S}\big)^{q(0)}+2^{v}$$

for any $v\leq 0$. The first claim can be reformulated as showing that

$$\int_{2^{v-1}}^{2^v}\big(\delta^{-\frac{1}{q(t)}}t^{-\theta}K(t,\frac{f}{S})\big)^{q(t)}\frac{dt}{t}\lesssim 1.$$

We need only to show that

$$\delta^{-\frac{1}{q(t)}}t^{-\theta}K(t,\frac{f}{S})\lesssim 1$$

for any $v\in\mathbb{N}$ and any $t\in[2^{v-1},2^v]$. From (1), the left-hand side is bounded by

$$\delta^{-\frac{1}{q(t)}}2^{-\theta v}K(2^v,\frac{f}{S}),$$

and from (3) we find that

$$\delta^{-\frac{1}{q(t)}}2^{-\theta v}K(2^v,\frac{f}{S}) \lesssim \delta^{-\frac{1}{q_\infty}}2^{-\theta v}K(2^v,\frac{f}{S}) \leq 1$$

for any $v \in \mathbb{N}$. Similarly we estimate the second claim. Hence the lemma is proved. □

Let (A_0,A_1) be a compatible couple. Let $\theta \in [0,1]$, $f \in (A_0,A_1)_{\theta,\infty}$ and we put $\alpha_v = K(2^v, f)$, $v \in \mathbb{Z}$. Then we have

$$\|f\|_{(A_0,A_1)_{\theta,\infty}} \approx \sup_{v\in\mathbb{Z}} 2^{v\theta}\alpha_v.$$

We present some important properties of the spaces $(A_0,A_1)_{\theta,q(\cdot)}$.

Theorem 3.5. *Let $\theta \in (0,1)$ and $q \in \mathcal{P}(\mathbb{R})$. Let (A_0,A_1) be a compatible couple of Banach spaces. Then $(A_0,A_1)_{\theta,q(\cdot)}$ is Banach space and*

$$K(s,f;A_0,A_1) \leq \gamma_{\theta,q^+,q^-}s^\theta\,\|f\|_{(A_0,A_1)_{\theta,q(\cdot)}}$$

for any $s > 0$. Moreover we have

$$A_0 \cap A_1 \hookrightarrow (A_0,A_1)_{\theta,q(\cdot)} \hookrightarrow A_0 + A_1.$$

Proof. Let $\{f_n\}_n$ be a sequence in $A_0 + A_1$ such that

$$\sum_{n=1}^{\infty} \|f_n\|_{(A_0,A_1)_{\theta,q(\cdot)}} < \infty.$$

Since $L^{q(\cdot)}((0,\infty),\frac{dt}{t})$ is a Banach space, the series $\sum_{n=1}^{\infty} t^{-\theta}K(t,f_n)$ converges in $L^{q(\cdot)}((0,\infty),\frac{dt}{t})$. Then clearly

$$\sum_{n=1}^{\infty} t^{-\theta}K(t,f_n) < \infty$$

for all $t > 0$. By the triangle inequality we have that

$$t^{-\theta}K(t,\sum_{n=1}^{\infty} f_n) \leq t^{-\theta}\sum_{n=1}^{\infty} K(t,f_n)$$

for all $t > 0$. Applying the $L^{q(\cdot)}((0,\infty),\frac{dt}{t})$-norm to each side, we obtain

$$\|\sum_{n=1}^{\infty} f_n\|_{(A_0,A_1)_{\theta,q(\cdot)}} \leq \sum_{n=1}^{\infty} \|f_n\|_{(A_0,A_1)_{\theta,q(\cdot)}} < \infty,$$

which ensure that $(A_0,A_1)_{\theta,q(\cdot)}$ is Banach space. By the property (1) we find that
$$\min(1,\frac{t}{s})K(s,f) \leq K(t,f), \quad s,t>0.$$
Therefore,
$$\big\| t^{-\theta}\min(1,\frac{t}{s})\big\|_{L^{q(\cdot)}((0,\infty),\frac{dt}{t})}K(s,f) \leq \|f\|_{(A_0,A_1)_{\theta,q(\cdot)}}.$$
Let us prove that
$$\big\| t^{-\theta}\min(1,\frac{t}{s})\big\|_{L^{q(\cdot)}((0,\infty),\frac{dt}{t})} \gtrsim s^{-\theta}. \tag{4}$$
We have
$$\big\| t^{-\theta}\min(1,\frac{t}{s})\big\|_{L^{q(\cdot)}((0,\infty),\frac{dt}{t})} \geq s^{-\theta}\big\| \big(\frac{t}{s}\big)^{1-\theta}\big\|_{L^{q(\cdot)}((0,s),\frac{dt}{t})},$$
and
$$\int_0^s \big(\frac{t}{s}\big)^{(1-\theta)q(t)}\frac{dt}{t} \geq \int_0^s \big(\frac{t}{s}\big)^{(1-\theta)q^+}\frac{dt}{t} = \frac{1}{(1-\theta)q^+}.$$
From Lemma 2.1, we find our claim (4). Therefore,
$$K(s,f) \leq \gamma_{\theta,q^+,q^-}s^{\theta}\|f\|_{(A_0,A_1)_{\theta,q(\cdot)}}$$
for any $s>0$. Taking $s=1$, we obtain
$$\|f\|_{A_0+A_1} \lesssim \|f\|_{(A_0,A_1)_{\theta,q(\cdot)}}.$$
Now since
$$K(t,f) \leq \min(1,t)\|f\|_{A_0\cap A_1},$$
we find that
$$\|f\|_{(A_0,A_1)_{\theta,q(\cdot)}} \lesssim \|f\|_{A_0\cap A_1}.$$
□

Definition 3.6. Let (A_0,A_1) and (B_0,B_1) be two compatible couples of Banach spaces and let T be a linear operator defined on A_0+A_1 and taking values in B_0+B_1. T is said be admissible with respect to the couples (A_0,A_1) and (B_0,B_1) if, for each $i=1,0$ the restriction of T to A_i maps A_i into B_i and furthermore is a bounded operator from A_i into B_i :
$$\|Tf\|_{B_i} \leq \|T\|_{L(A_i,B_i)}\|f\|_{A_i}, \quad f\in A_i.$$

Notice that every admissible operator T with respect to the couples (A_0,A_1) and (B_0,B_1) is bounded from A_0+A_1 into B_0+B_1.

Theorem 3.7. *Let $\theta \in (0,1)$ and $q \in \mathcal{P}(\mathbb{R})$. Let (A_0,A_1) and (B_0,B_1) be two compatible couples of Banach spaces and let T be admissible with respect to the couples (A_0,A_1) and (B_0,B_1). Then*

$$T : (A_0,A_1)_{\theta,q(\cdot)} \longrightarrow (B_0,B_1)_{\theta,q(\cdot)}$$

and

$$\|Tf\|_{(B_0,B_1)_{\theta,q(\cdot)}} \leq \max\Big(\|T\|_{L(A_0,B_0)}, \|T\|_{L(A_1,B_1)}\Big)\,\|f\|_{(A_0,A_1)_{\theta,q(\cdot)}}$$

for all $f \in (A_0,A_1)_{\theta,q(\cdot)}$.

Proof. Suppose that $T : (A_0,A_1) \longrightarrow (B_0,B_1)$. Then

$$\begin{aligned} K(t,Tf;B_0,B_1) &\leq \|T\|_{L(A_0,B_0)} K\Big(\frac{\|T\|_{L(A_1,B_1)}\,t}{\|T\|_{L(A_0,B_0)}}, f; A_0,A_1\Big) \\ &\leq \max\big(\|T\|_{L(A_0,B_0)}, \|T\|_{L(A_1,B_1)}\big) K(t,f;A_0,A_1), \end{aligned}$$

by the property (1). Multiplying by $t^{-\theta}$ and then applying the $L^{q(\cdot)}((0,\infty),\frac{dt}{t})$-norm to each side we obtain the desired estimate. □

Proposition 3.8. *Let $\theta \in (0,1)$. Let (A_0,A_1) be a compatible couples of Banach spaces.*
(i) Let $q,r \in \mathcal{P}(\mathbb{R})$ with $1 \leq q(\cdot) \leq r(\cdot) < \infty$. Then

$$(A_0,A_1)_{\theta,q(\cdot)} \hookrightarrow (A_0,A_1)_{\theta,r(\cdot)}.$$

and

$$(A_0,A_1)_{\theta,q(\cdot)} \hookrightarrow (A_0,A_1)_{\theta,\infty}$$

(ii) Let $q \in \mathcal{P}(\mathbb{R})$ be log-Hölder continuous both at the origin and at infinity with $q(0) = q_\infty$. Then

$$(A_0,A_1)_{\theta,q(\cdot)} = (A_1,A_0)_{1-\theta,q(\cdot)}.$$

(iii) Let $q,r \in \mathcal{P}(\mathbb{R})$ be log-Hö lder continuous both at the origin and at infinity with $q(0) = r(0)$ and $q_\infty = r_\infty$. Then

$$(A_0,A_1)_{\theta,q(\cdot)} = (A_0,A_1)_{\theta,r(\cdot)}.$$

(iv) If $A_1 \hookrightarrow A_0$, then

$$(A_0,A_1)_{\theta_1,q(\cdot)} \hookrightarrow (A_0,A_1)_{\theta,q(\cdot)} \quad \textit{if} \quad 0 < \theta \leq \theta_1 < 1.$$

(v) If $A_0 = A_1$, with equal norm, then

$$(A_0,A_1)_{\theta,q(\cdot)} = A_0.$$

Proof. We prove (i). From Theorem 3.5, we obtain

$$(A_0,A_1)_{\theta,q(\cdot)} \hookrightarrow (A_0,A_1)_{\theta,\infty} \quad \text{and} \quad K(s,f) \lesssim s^{\theta} \|f\|_{(A_0,A_1)_{\theta,q(\cdot)}}$$

for any $f \in (A_0,A_1)_{\theta,q(\cdot)}$, any $s>0$ with $\|f\|_{(A_0,A_1)_{\theta,q(\cdot)}} \neq 0$ and this implies that

$$\begin{aligned} &\int_0^\infty \big(t^{-\theta}K(t,\frac{f}{\|f\|_{(A_0,A_1)_{\theta,q(\cdot)}}})\big)^{r(t)}\frac{dt}{t} \\ \leq\ & \int_0^\infty \big(t^{-\theta}K(t,\frac{f}{\|f\|_{(A_0,A_1)_{\theta,q(\cdot)}}})\big)^{q(t)}\big(\sup_{t>0} t^{-\theta}K(t,\frac{f}{\|f\|_{(A_0,A_1)_{\theta,q(\cdot)}}})\big)^{r(t)-q(t)}\frac{dt}{t} \\ \lesssim\ & \int_0^\infty \big(t^{-\theta}K(t,\frac{f}{\|f\|_{(A_0,A_1)_{\theta,q(\cdot)}}})\big)^{q(t)}\frac{dt}{t}. \end{aligned}$$

The last term is bounded since

$$\Big\|t^{-\theta}K(t,\frac{f}{\|f\|_{(A_0,A_1)_{\theta,q(\cdot)}}})\Big\|_{L^{q(\cdot)}((0,\infty),\frac{dt}{t})} = 1,$$

which implies that,

$$\|f\|_{(A_0,A_1)_{\theta,r(\cdot)}} \lesssim \|f\|_{(A_0,A_1)_{\theta,q(\cdot)}}.$$

Hence the property (i) is proved. To prove (ii) we use Lemma 3.4, the fact that

$$K(t,f;A_0,A_1) = tK(t^{-1},f;A_1,A_0), \quad v>0,$$

and $q(0)=q_\infty$. The property (iii) follows by Lemma 3.4. Now if $A_1 \hookrightarrow A_0$ then we have $\|f\|_{A_0} \leq c\|f\|_{A_1}$ for any $f \in A_1$ and

$$K(t,f) = \|f\|_{A_0},$$

if $t>c$. Then

$$\big\|t^{-\theta}K(t,f)\big\|_{L^{q(\cdot)}((c,\infty),\frac{dt}{t})} \lesssim \|f\|_{A_0},$$

and

$$\big\|t^{-\theta}K(t,f)\big\|_{L^{q(\cdot)}((0,\infty),\frac{dt}{t})} \lesssim \big\|t^{-\theta}K(t,f)\big\|_{L^{q(\cdot)}((0,c),\frac{dt}{t})} + \|f\|_{A_0}.$$

Using the fact that

$$\|f\|_{A_0} \lesssim \big\|t^{-\theta_1}K(t,f)\big\|_{L^{q(\cdot)}((c,\infty),\frac{dt}{t})},$$

and $0<\theta\leq\theta_1<1$, we obtain

$$\|f\|_{(A_0,A_1)_{\theta,q(\cdot)}} \lesssim \|f\|_{(A_0,A_1)_{\theta_1,q(\cdot)}}.$$

So, the property (iv) is proved. Now the property (v) is immediate. The proof is complete. $\square$

4. The J–Method

Let (A_0,A_1) be a compatible couple. With $t>0$ fixed, put

$$J(t,f;A_0,A_1)=\inf_{f=f_0+f_1}\left(\|f_0\|_{A_0},t\,\|f_1\|_{A_1}\right),\quad f\in A_0\cap A_1.$$

Notice that $J(t,f;A_0,A_1)$ is an equivalent norm on $A_0\cap A_1$ for a given $t>0$. If there is no danger of confusion, we shall write $J(t,f)=J(t,f;A_0,A_1)$. For any $f\in A_0\cap A_1$, $J(t,f)$ is a positive, increasing and convex function of t, such that

$$J(t,f)\le\max(1,\frac{t}{s})J(s,f),\tag{5}$$

and

$$K(t,f)\le\min(1,\frac{t}{s})J(s,f).\tag{6}$$

Now we define the interpolation space constructed by the J–method.

Definition 4.1. Let $\theta\in(0,1)$ and $q\in\mathcal{P}(\mathbb{R})$. Let (A_0,A_1) be a compatible couple. The space $(A_0,A_1)_{\theta,q(\cdot),J}$ consists of all f in A_0+A_1 that are representable in the form

$$f=\int_0^\infty u(t)\frac{dt}{t}\tag{7}$$

where $u(t)$ is measurable with values in $A_0\cap A_1$ and

$$\|f\|_{(A_0,A_1)_{\theta,q(\cdot),J}}=\inf\left\|t^{-\theta}J(t,u(t))\right\|_{L^{q(\cdot)}((0,\infty),\frac{dt}{t})}<\infty,$$

where the infimum is taken over all u such that (7) holds.

Definition 4.2. Let $\theta\in(0,1)$. The space $(A_0,A_1)_{\theta,\infty,J}$ consists of all f in A_0+A_1 that are representable in the form (7), where $u(t)$ is measurable with values in $A_0\cap A_1$ and

$$\|f\|_{(A_0,A_1)_{\theta,\infty,J}}=\inf\sup_{t>0}t^{-\theta}J(t,u(t))<\infty,$$

where the infimum is taken over all u such that (7) holds.

Lemma 4.3. *Let (A_0,A_1) be a compatible couple and $f\in A_0+A_1$. Let $\theta\in(0,1)$ and $q\in\mathcal{P}(\mathbb{R})$ be log-Hölder continuous both at the origin and at infinity. Then $f\in(A_0,A_1)_{\theta,q(\cdot),J}$ if and only if there exist $u_v\in A_0\cap A_1$, $v\in\mathbb{Z}$, with*

$$f=\sum_{v=-\infty}^{\infty}u_v\quad\textit{convergence in } A_0\cap A_1,\tag{8}$$

and such that

$$\begin{aligned}&\|(J(2^v,u_v))_v\|_{\lambda^{\theta,q(0),q_\infty}}\\ =&\Big(\sum_{v=-\infty}^{0}2^{-v\theta q(0)}J(2^v,u_v)^{q(0)}\Big)^{\frac{1}{q(0)}}+\Big(\sum_{v=1}^{\infty}2^{-v\theta q_\infty}J(2^v,u_v)^{q_\infty}\Big)^{\frac{1}{q_\infty}}<\infty.\end{aligned}$$

Moreover

$$\|f\|_{(A_0,A_1)_{\theta,q(\cdot),J}}\approx\inf_{u_v}\|(J(2^v,u_v))_v\|_{\lambda^{\theta,q(0),q_\infty}},$$

where the infimum is extended over all sequences $(u_v)_v$ *satisfying* (8).

Proof. Let $f\in(A_0,A_1)_{\theta,q(\cdot),J}$. Then we have a representation

$$f=\int_0^\infty u(t)\frac{dt}{t},$$

where $u(t)$ is measurable with values in $A_0\cap A_1$ and

$$\|f\|_{(A_0,A_1)_{\theta,q(\cdot),J}}=\inf\big\|t^{-\theta}J(t,u(t))\big\|_{L^{q(\cdot)}((0,\infty),\frac{dt}{t})}<\infty.$$

We set

$$u_v=\int_{2^v}^{2^{v+1}}u(t)\frac{dt}{t},\quad v\in\mathbb{Z}.$$

Then we have

$$f=\sum_{v=-\infty}^{\infty}u_v.$$

Let us prove that

$$\begin{aligned}S(\{u_v\})&=\Big(\sum_{v=-\infty}^{0}2^{-v\theta q(0)}\alpha_v^{q(0)}\Big)^{\frac{1}{q(0)}}+\Big(\sum_{v=1}^{\infty}2^{-v\theta q_\infty}\alpha_v^{q_\infty}\Big)^{\frac{1}{q_\infty}}\\&\lesssim\big\|t^{-\theta}J(t,u(t))\big\|_{L^{q(\cdot)}((0,\infty),\frac{dt}{t})},\end{aligned}\tag{9}$$

with $\alpha_v=J(2^v,u_v)$, $v\in\mathbb{Z}$. We need only to prove that

$$\sum_{v=1}^{\infty}2^{-v\theta q_\infty}\Big(\frac{\alpha_v}{\|t^{-\theta}J(t,u(t))\|_{L^{q(\cdot)}((0,\infty),\frac{dt}{t})}}\Big)^{q_\infty}\lesssim 1,$$

and

$$\sum_{v=-\infty}^{0}2^{-v\theta q(0)}\Big(\frac{\alpha_v}{\|t^{-\theta}J(t,u(t))\|_{L^{q(\cdot)}((0,\infty),\frac{dt}{t})}}\Big)^{q(0)}\lesssim 1.$$

First let us prove that

$$2^{-v\theta q_\infty}\Big(\frac{\alpha_v}{\|t^{-\theta}J(t,u(t))\|_{L^{q(\cdot)}((0,\infty),\frac{dt}{t})}}\Big)^{q_\infty}$$
$$\leq \int_{2^{v-1}}^{2^v}\big(t^{-\theta}J(t,\frac{u(t)}{\|t^{-\theta}J(t,u(t))\|_{L^{q(\cdot)}((0,\infty),\frac{dt}{t})}})\big)^{q(t)}\frac{dt}{t}+2^{-v}=\delta$$

for any $v\in\mathbb{N}$. This claim can be reformulated as showing that

$$\Big(\delta^{-\frac{1}{q_\infty}}2^{-v\theta}\frac{\alpha_v}{\|t^{-\theta}J(t,u(t))\|_{L^{q(\cdot)}((0,\infty),\frac{dt}{t})}}\Big)^{q_\infty}\lesssim 1.$$

Using the property (5), we find that

$$\delta^{-\frac{1}{q_\infty}}2^{-v\theta}\frac{\alpha_v}{\|t^{-\theta}J(t,u(t))\|_{L^{q(\cdot)}((0,\infty),\frac{dt}{t})}}$$
$$\lesssim \int_{2^{v-1}}^{2^v}\delta^{-\frac{1}{q_\infty}}\tau^{-\theta}J(\tau,\frac{u(t)}{\|t^{-\theta}J(t,u(t))\|_{L^{q(\cdot)}((0,\infty),\frac{dt}{t})}})\frac{d\tau}{\tau}.$$

By Lemma 2.3,

$$\Big(\int_{2^{v-1}}^{2^v}\delta^{-\frac{1}{q_\infty}}\tau^{-\theta}J(\tau,\frac{u(t)}{\|t^{-\theta}J(t,u(t))\|_{L^{q(\cdot)}((0,\infty),\frac{dt}{t})}}\frac{d\tau}{\tau}\Big)^{q(t)}$$
$$\lesssim \int_{2^{v-1}}^{2^v}\delta^{-\frac{q(\tau)}{q_\infty}}\big(\tau^{-\theta}J(\tau,\frac{u(t)}{\|t^{-\theta}J(t,u(t))\|_{L^{q(\cdot)}((0,\infty),\frac{dt}{t})}})\big)^{q(\tau)}\frac{d\tau}{\tau}+1$$

for any $t\in[2^{v-1},2^v]$. Since, q is log-Hölder continuous at the infinity we find that

$$\delta^{-\frac{q(\tau)}{q_\infty}}\approx\delta^{-1},\quad \tau\in[2^{v-1},2^v],\quad v\in\mathbb{N}. \tag{10}$$

Therefore, from the definition of δ, we find that the last integral is dominated by a constant independent on $v\in\mathbb{N}$. Now, let us prove that

$$2^{-v\theta q(0)}\Big(\frac{\alpha_v}{\|t^{-\theta}J(t,u(t))\|_{L^{q(\cdot)}((0,\infty),\frac{dt}{t})}}\Big)^{q(0)}$$
$$\lesssim \int_{2^{v-1}}^{2^v}\big(t^{-\theta}J(t,\frac{u(t)}{\|t^{-\theta}J(t,u(t))\|_{L^{q(\cdot)}((0,\infty),\frac{dt}{t})}})\big)^{q(t)}\frac{dt}{t}+2^{v}=\delta$$

for any $v\leq 0$. This claim can be reformulated as showing that

$$\Big(\delta^{-\frac{1}{q(0)}}2^{-v\theta}\frac{\alpha_v}{\|t^{-\theta}J(t,u(t))\|_{L^{q(\cdot)}((0,\infty),\frac{dt}{t})}}\Big)^{q(0)}\lesssim 1.$$

The property (5), gives that

$$\begin{aligned}
&\delta^{-\frac{1}{q(0)}}2^{-v\theta}\frac{\alpha_v}{\|t^{-\theta}J(t,u(t))\|_{L^{q(\cdot)}((0,\infty),\frac{dt}{t})}}\\
\leq\ &\int_{2^{v-1}}^{2^v}\delta^{-\frac{1}{q(0)}}\tau^{-\theta}J(\tau,\frac{u(t)}{\|t^{-\theta}J(t,u(t))\|_{L^{q(\cdot)}((0,\infty),\frac{dt}{t})}})\frac{d\tau}{\tau}.
\end{aligned}$$

Again by Lemma 2.3,

$$\begin{aligned}
&\Big(\int_{2^{v-1}}^{2^v}\delta^{-\frac{1}{q(0)}}\tau^{-\theta}J(\tau,\frac{u(t)}{\|t^{-\theta}J(t,u(t))\|_{L^{q(\cdot)}((0,\infty),\frac{dt}{t})}})\frac{d\tau}{\tau}\Big)^{q(t)}\\
\lesssim\ &\int_{2^{v-1}}^{2^v}\delta^{-\frac{q(\tau)}{q(0)}}\big(\tau^{-\theta}J(\tau,\frac{u(t)}{\|t^{-\theta}J(t,u(t))\|_{L^{q(\cdot)}((0,\infty),\frac{dt}{t})}})\frac{d\tau}{\tau}\big)^{q(\tau)}\frac{d\tau}{\tau}+1
\end{aligned}$$

for any $t\in[2^{v-1},2^v]$ and any $v\leq 0$. We use the logarithmic decay condition at origin of q to show that

$$\delta^{-\frac{q(\tau)}{q(0)}}\approx\delta^{-1},\quad \tau\in[2^{v-1},2^v],\quad v\leq 0.$$

Therefore and from the definition of δ, we find that

$$\int_{2^{v-1}}^{2^v}\delta^{-\frac{q(\tau)}{q(0)}}\big(\tau^{-\theta}J(\tau,\frac{u(t)}{\|t^{-\theta}J(t,u(t))\|_{L^{q(\cdot)}((0,\infty),\frac{dt}{t})}})\big)^{q(\tau)}\frac{d\tau}{\tau}\lesssim 1.$$

Hence the left-hand side of (9) can be estimated by

$$c\int_0^\infty\big(t^{-\theta}J(t,\frac{u(t)}{\|t^{-\theta}J(t,u(t))\|_{L^{q(\cdot)}((0,\infty),\frac{dt}{t})}})\big)^{q(t)}\frac{dt}{t}+c.$$

The first term is bounded since

$$\Big\|t^{-\theta}J(t,\frac{u(t)}{\|t^{-\theta}J(t,u(t))\|_{L^{q(\cdot)}((0,\infty),\frac{dt}{t})}})\Big\|_{L^{q(\cdot)}((0,\infty),\frac{dt}{t})}=1.$$

Now in (9) taking the infimum, we conclude that

$$\inf_{u_v}\|(J(2^v,u_v))_v\|_{\lambda^{\theta,q(0),q_\infty}}\lesssim\|f\|_{(A_0,A_1)_{\theta,q(\cdot),J}}.$$

Conversely, assume that

$$f=\sum_{v=-\infty}^{\infty}u_v\quad\text{and}\quad S=S(\{u_v\})<\infty.$$

Let us prove that

$$\|f\|_{(A_0,A_1)_{\theta,q(\cdot),J}} \lesssim S. \tag{11}$$

Choose

$$u(t) = \frac{u_\nu}{\log 2}, \quad t \in [2^{\nu-1}, 2^\nu].$$

Then $f = \int_0^\infty u(t)\frac{dt}{t}$. This claim can be reformulated as showing that

$$\int_0^\infty \big(t^{-\theta} J(t, \frac{u(t)}{S})\big)^{q(t)} \frac{dt}{t} \lesssim 1.$$

Now our estimate clearly follows from the inequalities

$$\int_{2^{\nu-1}}^{2^\nu} \big(t^{-\theta} J(t, \frac{u(t)}{S})\big)^{q(t)} \frac{dt}{t} \lesssim 2^{-\nu\theta q_\infty} \big(\frac{\alpha_\nu}{S}\big)^{q_\infty} + 2^{-\nu} = \delta$$

for any $\nu \in \mathbb{N}$ and

$$\int_{2^{\nu-1}}^{2^\nu} \big(t^{-\theta} J(t, \frac{u(t)}{S})\big)^{q(t)} \frac{dt}{t} \lesssim 2^{-\nu\theta q(0)} \big(\frac{\alpha_\nu}{S}\big)^{q(0)} + 2^{\nu} = \delta$$

for any $\nu \leq 0$. The first claim can be reformulated as showing that

$$\int_{2^{\nu-1}}^{2^\nu} \big(\delta^{-\frac{1}{q(t)}} t^{-\theta} J(t, \frac{u(t)}{S})\big)^{q(t)} \frac{dt}{t} \lesssim 1.$$

We need only to show that

$$\delta^{-\frac{1}{q(t)}} t^{-\theta} J(t, \frac{u(t)}{S}) \lesssim 1$$

for any $\nu \in \mathbb{N}$ and any $t \in [2^{\nu-1}, 2^\nu]$. The left-hand side is bounded by

$$\delta^{-\frac{1}{q(t)}} 2^{-\theta\nu} J(2^\nu, \frac{u(t)}{S}).$$

From (10) we find that

$$\delta^{-\frac{1}{q(t)}} 2^{-\theta\nu} J(2^\nu, \frac{u(t)}{S}) \lesssim \delta^{-\frac{1}{q_\infty}} 2^{-\theta\nu} J(2^\nu, \frac{u_\nu}{S}) \leq 1$$

for any $\nu \in \mathbb{N}$ and any $t \in [2^{\nu-1}, 2^\nu]$. Similarly we estimate the second claim.

In (11) taking the infimum we conclude that

$$\|f\|_{(A_0,A_1)_{\theta,q(\cdot),J}} \lesssim \inf_{u_\nu} \|(J(2^\nu, u_\nu))_\nu\|_{\lambda^{\theta,q(0),q_\infty}}.$$

$\square$

We shall prove that the spaces generated by the K-and J-methods are the same.

Theorem 4.4. *Let* (A_0, A_1) *be a compatible couple. Let* $\theta \in (0,1)$ *and* $q \in \mathcal{P}(\mathbb{R})$ *be log-Hölder continuous both at the origin and at infinity. Then*

$$(A_0, A_1)_{\theta, q(\cdot), J} = (A_0, A_1)_{\theta, q(\cdot)},$$

with equivalence of norms.

Proof. Let $f \in (A_0, A_1)_{\theta, q(\cdot), J}$ with $f = \int_0^\infty u(s) \frac{ds}{s}$, where $u(s)$ is measurable with values in $A_0 \cap A_1$. By (6) we have

$$K(t, f) \leq \int_0^\infty K(t, u(s)) \frac{ds}{s} \leq \int_0^\infty \min(1, \frac{t}{s}) J(s, u(s)) \frac{ds}{s}.$$

Applying Hardy inequality, Lemma 2.4, we get

$$\|f\|_{(A_0, A_1)_{\theta, q(\cdot)}} \lesssim \|f\|_{(A_0, A_1)_{\theta, q(\cdot), J}}.$$

For the converse inequality, Lemma 3.3.2 of [5], and using Theorem 3.5, implies the existence of a representation

$$f = \sum_{v=-\infty}^{\infty} u_v,$$

such that

$$J(2^v, u_v) \leq (\gamma + \varepsilon) K(2^v, f)$$

for any $v \in \mathbb{Z}$, $\varepsilon > 0$ and γ is a universal constant less than or equal 3. By Lemmas 3.4 and 4.3 we get

$$\|f\|_{(A_0, A_1)_{\theta, q(\cdot), J}} \lesssim \|f\|_{(A_0, A_1)_{\theta, q(\cdot)}}.$$

This completes the proof of this theorem. $\square$

Theorem 4.5. *Let* (A_0, A_1) *be a compatible couple. Let* $\theta \in (0,1)$ *and* $q \in \mathcal{P}(\mathbb{R})$ *be log-Hölder continuous both at the origin and at infinity. Then* $A_0 \cap A_1$ *is dense in* $(A_0, A_1)_{\theta, q(\cdot)}$.

Proof. Let $f \in (A_0, A_1)_{\theta, q(\cdot)}$. From Theorem 4.4 we have

$$f = \sum_{v=-\infty}^{\infty} u_v,$$

where u_ν, $\nu \in \mathbb{Z}$ is measurable with values in $A_0 \cap A_1$ and

$$\|(J(2^\nu, u_\nu))_\nu\|_{\lambda^{\theta,q(0),q_\infty}} < \infty.$$

Then

$$\begin{aligned} &\Big\| f - \sum_{|\nu|\le N} u_\nu \Big\|_{(A_0,A_1)_{\theta,q(\cdot)}} \\ \le\ & \Big(\sum_{\nu=N}^{\infty} 2^{-\nu\theta q_\infty} J(2^\nu, u_\nu)^{q_\infty}\Big)^{\frac{1}{q_\infty}} + \Big(\sum_{\nu=-\infty}^{-N} 2^{-\nu\theta q(0)} J(2^\nu, u_\nu)^{q(0)}\Big)^{\frac{1}{q(0)}}. \end{aligned}$$

Therefore,

$$\Big\| f - \sum_{|\nu|\le N} u_\nu \Big\|_{(A_0,A_1)_{\theta,q(\cdot)}},$$

which tends to zero if $N \longrightarrow \infty$. □

Definition 4.6. Let $\theta \in [0,1]$. Let (A_0,A_1) be a compatible couple of normed vector spaces. Suppose that X is an intermediate space with respect to (A_0,A_1). Then we say that
(i) X is of class $\mathscr{C}_K(\theta;A_0,A_1)$ if $K(t,f;A_0,A_1) \le Ct^\theta \|f\|_X, \quad f \in X$;
(ii) X is of class $\mathscr{C}_J(\theta;A_0,A_1)$ if $\|f\|_X \le Ct^{-\theta} J(t,f;A_0,A_1), \quad f \in A_0 \cap A_1$.
(iii) We say that X is of class $\mathscr{C}(\theta;A_0,A_1)$ if X is of class $\mathscr{C}_K(\theta;A_0,A_1)$ and of class $\mathscr{C}_J(\theta;A_0,A_1)$.

Let $q \in \mathcal{P}(\mathbb{R})$. From [5, Theorem 3.5.2], Theorem 3.5 and Proposition 3.8 we see that $(A_0,A_1)_{\theta,q(\cdot)}$ is of class $\mathscr{C}(\theta;A_0,A_1)$ if $\theta \in (0,1)$.

We are now ready to prove the reiteration theorem, which is one of the most important general results in interpolation theory.

Theorem 4.7. *Let $q \in \mathcal{P}(\mathbb{R})$ be log-Hölder continuous both at the origin and at infinity. Let (A_0,A_1) and (X_0,X_1) be two compatible couples of normed linear spaces, and assume that X_i ($i = 0,1$) are complete and of class $\mathscr{C}(\theta_i;A_0,A_1)$, where $\theta_0,\theta_1 \in [0,1]$ and $\theta_0 \ne \theta_1$. Put*

$$\theta = (1-\eta)\theta_0 + \eta\theta_1, \quad \eta \in (0,1).$$

Then

$$(A_0,A_1)_{\theta,q(\cdot)} = (X_0,X_1)_{\eta,q(\cdot)}$$

with equivalence of norms. In particular, if $\theta_0,\theta_1 \in (0,1)$, $q_0,q_1 \in \mathcal{P}(\mathbb{R})$ are log-Hölder continuous both at the origin and at infinity and $(A_0,A_1)_{\theta_i,q_i(\cdot)}$ are complete then

$$\big((A_0,A_1)_{\theta_0,q_0(\cdot)}, (A_0,A_1)_{\theta_1,q_1(\cdot)}\big)_{\eta,q(\cdot)} = (A_0,A_1)_{\theta,q(\cdot)}$$

where

$$\frac{1}{q(\cdot)} = \frac{\theta_0}{q_0(\cdot)} + \frac{\theta_1}{q_1(\cdot)}.$$

Proof. We will do the proof in two steps.

Step 1. Let us prove that

$$(X_0, X_1)_{\eta, q(\cdot)} \hookrightarrow (A_0, A_1)_{\theta, q(\cdot)}. \tag{12}$$

Let $f \in (X_0, X_1)_{\eta, q(\cdot)}$. Then

$$f = f_0 + f_1, \quad f_0 \in X_0, f_1 \in X_1.$$

Since X_i $(i = 0, 1)$ are of class $\mathscr{C}(\theta_i; A_0, A_1)$ we have

$$\begin{aligned} K(t, f; A_0, A_1) &\leq K(t, f_0; A_0, A_1) + K(t, f_1; A_0, A_1) \leq c(t^{\theta_0} \|f_0\|_{X_0} + t^{\theta_1} \|f_1\|_{X_1}) \\ &\leq ct^{\theta_0} K(t^{\theta_1 - \theta_0}, f; X_0, X_1). \end{aligned}$$

Therefore, from Lemma 3.4, we get

$$\begin{aligned} \|f\|_{(A_0,A_1)_{\theta,q(\cdot)}} &\lesssim \Big(\int_0^1 t^{(\theta_0 - \theta) q(0)} K(t^{\theta_1 - \theta_0}, f; X_0, X_1)^{q(0)} \frac{dt}{t} \Big)^{\frac{1}{q(0)}} \\ &\quad + \Big(\int_1^\infty t^{(\theta_0 - \theta) q_\infty} K(t^{\theta_1 - \theta_0}, f; X_0, X_1)^{q_\infty} \frac{dt}{t} \Big)^{\frac{1}{q_\infty}}. \end{aligned}$$

Putting $s = t^{\theta_1 - \theta_0}$ and observing that $\eta = \frac{\theta - \theta_0}{\theta_1 - \theta_0}$ we find that

$$\|f\|_{(A_0,A_1)_{\theta,q(\cdot)}} \lesssim \|f\|_{(X_0,X_1)_{\eta,q(\cdot)}},$$

which gives (12).

Step 2. Let us prove that

$$(A_0, A_1)_{\theta, q(\cdot)} \hookrightarrow (X_0, X_1)_{\eta, q(\cdot)}. \tag{13}$$

Assume that $f \in (A_0, A_1)_{\theta, q(\cdot)}$. We choose a representation

$$f = \sum_{v=-\infty}^{\infty} u_v,$$

where u_v, $v \in \mathbb{Z}$ is measurable with values in $A_0 \cap A_1$ and

$$\|(J(2^v, u_v))_v\|_{\lambda^{\theta, q(0), q_\infty}} < \infty.$$

Applying (6), and that X_i $(i=0,1)$ are of class $\mathscr{C}(\theta_i;A_0,A_1)$ we get for any $j\in\mathbb{Z}$,

$$\begin{aligned}
& 2^{(\theta_0-\theta)j}K(2^{(\theta_1-\theta_0)j},f;X_0,X_1) \\
\leq\ & 2^{(\theta_0-\theta)j}\sum_{v=-\infty}^{\infty}K(2^{(\theta_1-\theta_0)j},u_v;X_0,X_1) \\
\leq\ & 2^{(\theta_0-\theta)j}\sum_{v=-\infty}^{\infty}\min\left(1,2^{(j-v)(\theta_1-\theta_0)}\right)J(2^{v(\theta_1-\theta_0)},u_v;X_0,X_1) \\
\leq\ & 2^{-\theta j}\sum_{v=-\infty}^{\infty}\min\left(2^{(j-v)\theta_0},2^{(j-v)\theta_1}\right)J(2^v,u_v;A_0,A_1).
\end{aligned}$$

The last term can be rewritten us

$$\sum_{v=-\infty}^{j}2^{(j-v)(\theta_0-\theta)}2^{-v\theta}J(2^v,u_v;A_0,A_1)+\sum_{v=j+1}^{\infty}2^{(j-v)(\theta_1-\theta)}2^{-v\theta}J(2^v,u_v;A_0,A_1) \tag{14}$$

for any $j\in\mathbb{Z}$. We treat the case $j\geq 0$. The first sum can be rewritten us

$$\begin{aligned}
& \sum_{v=-\infty}^{0}2^{(j-v)(\theta_0-\theta)}2^{-v\theta}J(2^v,u_v;A_0,A_1)+\sum_{v=1}^{j}2^{(j-v)(\theta_0-\theta)}2^{-v\theta}J(2^v,u_v;A_0,A_1) \\
\lesssim\ & 2^{j(\theta_0-\theta)}\sup_{v\leq 0}(2^{-v\theta}J(2^v,u_v;A_0,A_1))+\sum_{v=1}^{j}2^{(j-v)(\theta_0-\theta)}2^{-v\theta}J(2^v,u_v;A_0,A_1).
\end{aligned}$$

Applying Lemma 2.2 we get

$$\left\|\left(2^{(\theta_0-\theta)j}K(2^{(\theta_1-\theta_0)j},f;X_0,X_1)\right)_{j\geq 1}\right\|_{\lambda^{\theta,q(0),q_\infty}}\lesssim\|(J(2^v,u_v))_v\|_{\lambda^{\theta,q(0),q_\infty}}.$$

Now if $j\leq 0$, the second sum of (14) can be rewritten us

$$\begin{aligned}
& \sum_{v=j+1}^{0}2^{(j-v)(\theta_1-\theta)}2^{-v\theta}J(2^v,u_v;A_0,A_1)+\sum_{v=1}^{\infty}2^{(j-v)(\theta_1-\theta)}2^{-v\theta}J(2^v,u_v;A_0,A_1) \\
\leq\ & \sum_{v=j+1}^{0}2^{(j-v)(\theta_1-\theta)}2^{-v\theta}J(2^v,u_v;A_0,A_1)+2^{j(\theta_1-\theta)}\sup_{v\geq 1}\left(2^{-v\theta}J(2^v,u_v;A_0,A_1)\right).
\end{aligned}$$

Applying again Lemma 2.2 we get

$$\left\|\left(2^{(\theta_0-\theta)j}K(2^{(\theta_1-\theta_0)j},f;X_0,X_1)\right)_{j\leq 0}\right\|_{\lambda^{\theta,q(0),q_\infty}}\lesssim\|(J(2^v,u_v))_v\|_{\lambda^{\theta,q(0),q_\infty}}.$$

This prove the embedding (13) by taking the infimum in view of Theorem 4.3 and the fact that

$$\|f\|_{(X_0,X_1)_{\eta,q(\cdot)}} \approx \Big(\sum_{j=-\infty}^{0} 2^{(\theta_0-\theta)jq(0)}\alpha_j^{q(0)}\Big)^{\frac{1}{q(0)}} + \Big(\sum_{j=1}^{\infty} 2^{(\theta_0-\theta)jq_\infty}\alpha_j^{q_\infty}\Big)^{\frac{1}{q_\infty}},$$

where

$$\alpha_j = K(2^{(\theta_1-\theta_0)j}, f; X_0, X_1), \quad j \in \mathbb{Z}.$$

This completes the proof of Theorem 4.7. □

5. Application

In this section, we give a simple application of the results of the previous sections. We will present various real interpolation formulas in Besov spaces with variable indices. The symbol $\mathcal{S}(\mathbb{R}^n)$ is used in place of the set of all Schwartz functions on $\mathbb{R}^n$. We denote by $\mathcal{S}'(\mathbb{R}^n)$ the dual space of all tempered distributions on $\mathbb{R}^n$. The Fourier transform of a Schwartz function f is denoted by $\mathcal{F}f$. To define the variable Besov spaces, we first need the concept of a smooth dyadic resolution of unity. Let Ψ be a function in $\mathcal{S}(\mathbb{R}^n)$ satisfying $\Psi(x) = 1$ for $|x| \leq 1$ and $\Psi(x) = 0$ for $|x| \geq 2$. We define φ_0 and φ_1 by $\mathcal{F}\varphi_0(x) = \Psi(x)$, $\mathcal{F}\varphi_1(x) = \Psi(x) - \Psi(2x)$ and

$$\mathcal{F}\varphi_j(x) = \mathcal{F}\varphi_1(2^{-j}x) \quad \text{for} \quad j = 2,3,\ldots.$$

Then $\{\mathcal{F}\varphi_j\}_{j\in\mathbb{N}_0}$ is a smooth dyadic resolution of unity, $\sum_{j=0}^{\infty}\mathcal{F}\varphi_j(x) = 1$ for all $x \in \mathbb{R}^n$. Thus we obtain the Littlewood-Paley decomposition $f = \sum_{j=0}^{\infty}\varphi_j * f$ of all $f \in \mathcal{S}'(\mathbb{R}^n)$ (convergence in $\mathcal{S}'(\mathbb{R}^n)$).

Let $p,q \in \mathcal{P}(\mathbb{R}^n)$. The mixed Lebesgue-sequence space $\ell_>^{q(\cdot)}(L^{p(\cdot)})$ is defined on sequences of $L^{p(\cdot)}$-functions by the modular

$$\rho_{\ell_>^{q(\cdot)}(L^{p(\cdot)})}((f_v)_v) = \sum_{v=1}^{\infty} \inf\Big\{\lambda_v > 0 : \rho_{p(\cdot)}\Big(\frac{f_v}{\lambda_v^{1/q(\cdot)}}\Big) \leq 1\Big\}.$$

The (quasi)-norm is defined from this as usual:

$$\|(f_v)_v\|_{\ell_>^{q(\cdot)}(L^{p(\cdot)})} = \inf\Big\{\mu > 0 : \rho_{\ell^{q(\cdot)}(L^{p(\cdot)})}\Big(\frac{1}{\mu}(f_v)_v\Big) \leq 1\Big\}. \tag{15}$$

If $q^+ < \infty$, then we can replace (15) by the simpler expression $\rho_{\ell_>^{q(\cdot)}(L^{p(\cdot)})}((f_v)_v) = \sum_{v=1}^{\infty} \big\||f_v|^{q(\cdot)}\big\|_{\frac{p(\cdot)}{q(\cdot)}}$. The case $p := \infty$ can be included by replacing the last modular by $\rho_{\ell_>^{q(\cdot)}(L^\infty)}((f_v)_v) = \sum_{v=1}^{\infty} \big\| |f_v|^{q(\cdot)} \big\|_\infty$.

We define the following class of variable exponents $\mathcal{P}^{\log}(\mathbb{R}^n) := \left\{p \in \mathcal{P} : \frac{1}{p} \in C^{\log}\right\}$, were introduced in [9, Section 2]. We define $\frac{1}{p_\infty} := \lim_{|x|\to\infty} \frac{1}{p(x)}$ and we use the convention $\frac{1}{\infty} = 0$. Note that although $\frac{1}{p}$ is bounded, the variable exponent p itself can be unbounded.

We state the definition of the spaces $B^{s(\cdot)}_{p(\cdot),q(\cdot)}$, which introduced and investigated in [1].

Definition 5.1. *Let $\left\{\mathcal{F}\varphi_j\right\}_{j\in\mathbb{N}_0}$ be a resolution of unity,* $s : \mathbb{R}^n \to \mathbb{R}$ and $p, q \in \mathcal{P}(\mathbb{R}^n)$*. The Besov space $B^{s(\cdot)}_{p(\cdot),q(\cdot)}$ consists of all distributions $f \in \mathcal{S}'(\mathbb{R}^n)$ such that*

$$\|f\|_{B^{s(\cdot)}_{p(\cdot),q(\cdot)}} = \left\|(2^{js(\cdot)}\varphi_j * f)_j\right\|_{\ell^{q(\cdot)}_{>}(L^{p(\cdot)})} < \infty.$$

Taking $s \in \mathbb{R}$ and $q \in (0,\infty]$ as constants we derive the spaces $B^s_{p(\cdot),q}$ studied by Xu in [22]. We refer the reader to the recent papers [10], [13],[14] and [15] for further details, historical remarks and more references on these function spaces. For any $p, q \in \mathcal{P}^{\log}_0(\mathbb{R}^n)$ and $s \in C^{\log}_{\mathrm{loc}}(\mathbb{R}^n)$, the space $B^{s(\cdot)}_{p(\cdot),q(\cdot)}$ does not depend on the chosen smooth dyadic resolution of unity $\{\mathcal{F}\varphi_j\}_{j\in\mathbb{N}_0}$ (in the sense of equivalent quasi-norms) and

$$\mathcal{S}(\mathbb{R}^n) \hookrightarrow B^{s(\cdot)}_{p(\cdot),q(\cdot)} \hookrightarrow \mathcal{S}'(\mathbb{R}^n).$$

Moreover, if p, q, s are constants, we re-obtain the usual Besov spaces $B^s_{p,q}$, studied in detail in [16], [17], [19], [20] and [21].

Applying Lemma 3.4 and using the same arguments of [2, Theorem 3.1] we obtain.

Theorem 5.2. *Let $\theta \in (0,1)$ and $q \in \mathcal{P}(\mathbb{R})$ be log-Hölder continuous both at the origin and at infinity with $q(0) = q_\infty$. Let $p, q_0, q_1 \in \mathcal{P}^{\log}(\mathbb{R}^n)$ and $\alpha_0, \alpha_1 \in C^{\log}_{\mathrm{loc}}$. If $0 \neq \alpha_0 - \alpha_1$ is constant, then*

$$(B^{\alpha_0(\cdot)}_{p(\cdot),q_0(\cdot)}, B^{\alpha_1(\cdot)}_{p(\cdot),q_1(\cdot)})_{\theta,q(\cdot)} = B^{\alpha(\cdot)}_{p(\cdot),q(0)}$$

with $\alpha(\cdot) = (1-\theta)\alpha_0(\cdot) + \theta\alpha_1(\cdot)$. Moerover

$$(B^{\alpha(\cdot)}_{p(\cdot),r_0}, B^{\alpha(\cdot)}_{p(\cdot),r_1})_{\theta,q(\cdot)} = B^{\alpha(\cdot)}_{p(\cdot),q(0)},$$

with $r_0, r_1 \in [1,\infty]$ and

$$\frac{1}{q(0)} = \frac{1-\theta}{r_0} + \frac{\theta}{r_1}.$$

Now we present some interpolation results in variable exponent Lorentz spaces $\mathcal{L}^{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ introduced by [12].

Definition 5.3. If f is a measurable function on $\mathbb{R}^n$, then we define the non–increasing rearrangement of f through

$$f^*(t) = \sup\{\lambda > 0 : m_f(\lambda) > t\}$$

where m_f is the distribution function of f.

Definition 5.4. Let $p, q \in \mathcal{P}(\mathbb{R})$. By $\mathcal{L}^{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ we denote the space of functions f on $\mathbb{R}^n$ such that

$$\|f\|_{\mathcal{L}^{p(\cdot),q(\cdot)}(\mathbb{R}^n)} = \left\| t^{\frac{1}{p(t)}-\frac{1}{q(t)}} f^*(t) \right\|_{L^{q(\cdot)}([0,\infty))} < \infty.$$

We refer to the recent paper [12] for further details on these scales of spaces. We present an equivalent quasi-norm for the space $\mathcal{L}^{p(\cdot),q(\cdot)}(\mathbb{R}^n)$, where the proof is quite similar to that for Lemma 3.4.

Lemma 5.5. *Let $p, q \in \mathcal{P}(\mathbb{R})$ be log-Hölder continuous both at the origin and at infinity. Then*

$$\|f\|_{\mathcal{L}^{p(\cdot),q(\cdot)}(\mathbb{R}^n)} \approx \Big(\sum_{v=-\infty}^{0} 2^{-v\frac{q(0)}{p(0)}} (f^*(2^v))^{q(0)} \Big)^{\frac{1}{q(0)}} + \Big(\sum_{v=1}^{\infty} 2^{-v\frac{q_\infty}{p_\infty}} (f^*(2^v))^{q_\infty} \Big)^{\frac{1}{q_\infty}}.$$

Moreover,

$$\|f\|_{\mathcal{L}^{p(\cdot),q(\cdot)}(\mathbb{R}^n)} \approx \Big(\int_0^1 t^{-\frac{q(0)}{p(0)}} (f^*(t))^{q(0)} \frac{dt}{t} \Big)^{\frac{1}{q(0)}} + \Big(\int_1^\infty t^{-\frac{q_\infty}{p_\infty}} (f^*(t))^{q_\infty} \frac{dt}{t} \Big)^{\frac{1}{q_\infty}}.$$

Applying this lemma and [5, Theorem 5.2.1] we obtain.

Theorem 5.6. *Let $\theta \in (0,1)$ and $q \in \mathcal{P}(\mathbb{R})$ be log-Hölder continuous both at the origin and at infinity with $q(0) = q_\infty$. Then*

$$(L^1, L_\infty)_{\theta,q(\cdot)} = \mathcal{L}^{p,q(\cdot)}(\mathbb{R}^n), \quad p = \frac{1}{1-\theta}.$$

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